Towards non-trivial scalar theories in d≥4 space-time dimensions

1983 J. Phys. A: Math. Gen. 16 2521

(http://iopscience.iop.org/0305-4470/16/11/022)

J. Phys. A: Math. Gen. 16 (1983) 2521–2540. Printed in Great Britain

Towards non-trivial scalar theories in $d \geqslant 4$ space–time dimensions

R J Rivers

Imperial College of Science and Technology, Blackett Laboratory, Prince Consort Road, London SW7 2BZ, UK

Received 6 December 1982, in final form 1 February 1983

Abstract. We show, in the large-N limit, how *non-canonical* quantisation of an O(N)-invariant (pseudo) free scalar theory in $d \geqslant 4$ dimensions can give non-trivial behaviour. Implications for recent results on the triviality of canonical scalar theories in $d \geqslant 4$ dimensions are discussed.

1. Introduction

It has recently been shown (Aizenmann 1981, Frohlich 1982) that a $\lambda\varphi^4$ theory of a single scalar field is trivial in $d \geqslant 4$ space–time dimensions, and almost certainly trivial in $d = 4$ dimensions, when quantised *canonically*.

In response to this, Klauder (1981a, b) has argued that it is possible for a scale-covariant *non-canonical* quantisation of the same theory to render it *non-trivial*, since the bounds that enforce triviality of canonical scalar theories are weakened in scale-covariant quantisation. The basic idea (see Klauder 1979 and references therein) is that a $\lambda\varphi^4$ self-interaction provides a *discontinuous perturbation* for a conventionally non-superrenormalisable theory†. On switching off the φ^4 self-interaction, for such a theory in $d \geqslant 4$ dimensions, we recover the so called 'pseudofree' theory. If this pseudofree theory is non-trivial the $\lambda\varphi^4$ theory built upon it will be non-trivial.

From this viewpoint the problem of demonstrating the existence of non-trivial scalar theories in $d \geqslant 4$ dimensions reduces to the problem of demonstrating the existence of a *non-trivial* pseudofree scalar theory in these dimensions.

We have a more modest aim. The numerical work of Klauder (1981a, b) has made it plausible that non-trivial theories exist in $d \geqslant 4$ dimensions. The difficulty is to understand, in an analytic way, how a non-trivial pseudofree theory can arise.

In this paper we examine the large-N limit of an O(N)-invariant pseudofree theory for non-triviality in $d \geqslant 4$ dimensions. We have already seen (Ebbutt and Rivers 1982c) that, in general, the large-N pseudofree theory is *free*. We are concerned with exceptions to this general rule. We shall argue that, if non-trivial scalar theories do exist in $d \geqslant 4$ dimensions, then the large-N non-trivial pseudofree theories (compatible with Klauder 1981a) provide a good basis for our understanding of them.

This paper is organised as follows. The next section reviews Klauder's approach. More recapitulation takes place in the following section where we rederive the effective

† The case of $d = 4$ dimensions is seen as the boundary dimension of non-renormalisability, rather than of renormalisability.

potential $\mathcal{V}(\varphi)$ for the large-N limit of the O(N)-invariant scalar *pseudofree* theory first given by Ebbutt and Rivers (1982c). In the sections following we analyse $\mathcal{V}$ to see under what conditions the large-N theory becomes non-trivial. Having established non-triviality, we then revert to the effective action to provide a dynamical description of the self-interaction. The final section is a summary and analysis of our results.

2. Non-canonical scale-covariant quantisation

The path-integral formalism provides the simplest way to understand how discontinuous perturbations arise. The assumption (Klauder 1979) is that those theories that would be conventionally classified as non-superrenormalisable are not permitted the full range of field configuration histories available to a free field. That is, a partial 'hard-core' has to be accommodated, via a change in the functional measure.

These arguments are supported by

(i) analogies with singular potentials in quantum mechanics;

(ii) analogies with noise theory;

(iii) explicit models like the independent-value and ultra-local models which, although dynamically uninteresting, are technically non-trivial

(see Klauder (1978) for a non-technical summary of these approaches).

Measures in path integrals cannot be changed at will. The only alternatives to the orthodox translation-invariant measures (implied by, and implying, the canonical commutation relations) are scale-covariant measures (implied by, and implying, affine commutation relations).

Thus, we are led to consider the Euclidean pseudofree theory for a single scalar field φ with generating functional† ($\mathrm{d}\boldsymbol{x} \equiv \mathrm{d}^d x$)

$$Z'[j] = \int \mathscr{D}'[\varphi] \exp\left(-\hbar^{-1} \int \mathrm{d}\boldsymbol{x}[\tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2}m_0^2\varphi^2 - j\varphi]\right) \tag{2.1}$$

where $\mathscr{D}'[\varphi]$ is a scale-covariant measure satisfying

$$\mathscr{D}'[\Lambda\varphi] = F[\Lambda]\mathscr{D}'[\varphi] \qquad (\Lambda(x) > 0\ \forall x). \tag{2.2}$$

In terms of the translation-invariant measure $\mathscr{D}[\varphi]$, satisfying

$$\mathscr{D}[\varphi + \Lambda] = \mathscr{D}[\varphi] \qquad \text{for arbitrary } \Lambda(x) \tag{2.3}$$

we can formally express $\mathscr{D}'[\varphi]$ as

$$\mathscr{D}'[\varphi] = \frac{\mathscr{D}[\varphi]}{\Pi_x |\varphi(x)|^\beta} \qquad 0 < \beta \leq 1 \tag{2.4}$$

$$= \mathscr{D}[\varphi] \exp\left(-\tfrac{1}{2}\beta\delta(0) \int \mathrm{d}\boldsymbol{x} \ln \varphi(x)^2\right). \tag{2.5}$$

Different values of β correspond to different ways of quantising the theory.

We should not expect $Z'[j]$ of (2.1) to describe a *non-trivial* pseudofree theory for each value of β. In the precontinuous lattice formulation of (2.1) discussed by

† We stress that we always interpret a pseudofree theory as a limit of a self-interacting non-superrenormalisable theory as the interaction is switched off, and not as a free theory quantised in a peculiar way.

Klauder (1981a, b), the non-triviality of the pseudofree theory is equated to hyperscaling conservation. In order that the appropriate critical index be identically zero, it is argued that β has to be tuned to a unique value at the critical analogue temperature of the lattice theory. Otherwise hyperscaling violation occurs and we have a trivial theory.

In the absence of direct numerical integration we are obliged to rely on the uncertainties of Padé-like manipulations of high-temperature series. Qualitatively, to get a non-trivial pseudofree theory in $d=4$ dimensions we seem to need a value of β compatible with zero (Klauder 1981a, b). However, it is difficult to see what is happening in numerical calculations of this kind. In particular, the nature of the φ–φ force, that springs into existence when β is carefully tuned, is unclear.

In the remainder of this paper we shall show how, and why, an analytic approximation to Z' gives rise to unique values of β for non-trivial pseudofree theories.

3. The large-N limit of the O(N) pseudofree theory

In practice we can only perform Gaussian integration and, as it stands after the insertion of (2.4), $Z'[j]$ does not lend itself naturally to expansion about a Gaussian. In order to get a tractable analytic theory it is necessary to generalise the pseudofree theory given above to an O(N)-invariant theory with N fields φ_i, with generating functional

$$Z'[\boldsymbol{j}]=\int \mathscr{D}'[\boldsymbol{\varphi}]\exp\Big(-\hbar^{-1}\int \mathrm{d}\boldsymbol{x}[\tfrac{1}{2}(\nabla\boldsymbol{\varphi})^2+\tfrac{1}{2}m_0^2\boldsymbol{\varphi}^2-\boldsymbol{j}\cdot\boldsymbol{\varphi}]\Big). \tag{3.1}$$

The measure $\mathscr{D}'[\boldsymbol{\varphi}]$ is invariant under global O(N) transformations $\boldsymbol{\varphi}\to\boldsymbol{R\varphi}$ and covariant under scale transformations $\boldsymbol{\varphi}(x)\to\Lambda(x)(\boldsymbol{R\varphi})(x)$, $\Lambda(x)>0$.

It follows that $\mathscr{D}'[\boldsymbol{\varphi}]$ can be expressed as (the generalisation of (2.5))

$$\mathscr{D}'[\boldsymbol{\varphi}]=\prod_1^N\mathscr{D}[\varphi_i]\exp\Big(-\tfrac{1}{2}N\beta\delta(0)\int \mathrm{d}\boldsymbol{x}\,\ln(\boldsymbol{\varphi}(x)^2/N)\Big). \tag{3.2}$$

We expect the dominant regions of integration in (3.1) to come from $\boldsymbol{\varphi}(x)^2=\mathrm{O}(N)$. To make this more explicit we rewrite $Z'[\boldsymbol{j}]$ as

$$\begin{aligned}Z'[\boldsymbol{j}]&=\int\prod_1^N\mathscr{D}[\varphi_i]\mathscr{D}[\rho][\delta(\boldsymbol{\varphi}^2-N\rho)]\\&\quad\times\exp\Big[-\hbar^{-1}\Big(\int \mathrm{d}\boldsymbol{x}[\tfrac{1}{2}\boldsymbol{\varphi}\cdot(-\nabla^2+m_0^2)\boldsymbol{\varphi}-\boldsymbol{j}\cdot\boldsymbol{\varphi}]+\tfrac{1}{2}\hbar N\beta\delta(0)\int \mathrm{d}\boldsymbol{x}\,\ln\rho(x)\Big)\Big] \qquad (3.3)\\&=\int\prod_1^N\mathscr{D}[\varphi_i]\mathscr{D}[\rho]\mathscr{D}[\alpha]\exp\Big[-\hbar^{-1}\Big(\int \mathrm{d}\boldsymbol{x}[\tfrac{1}{2}\boldsymbol{\varphi}\cdot(-\nabla^2+m_0^2+\mathrm{i}\alpha)\boldsymbol{\varphi}-\boldsymbol{j}\cdot\boldsymbol{\varphi}]\\&\quad+\tfrac{1}{2}\mathrm{i}N\int \mathrm{d}\boldsymbol{x}\,\alpha\rho+\tfrac{1}{2}\hbar N\beta\delta(0)\int \mathrm{d}\boldsymbol{x}\,\ln\rho\Big)\Big]. \qquad (3.4)\end{aligned}$$

Defining $\chi=m_0^2+\mathrm{i}\alpha$ and integrating over the φ_i fields gives

$$Z'[\boldsymbol{j}]=\int\mathscr{D}[\rho]\mathscr{D}[\chi]\exp(-\hbar^{-1}N\mathfrak{A}[\rho,\chi,\boldsymbol{j}]) \tag{3.5}$$

where

$$\mathfrak{A}[\rho, \chi, \boldsymbol{j}] = -\tfrac{1}{2}N^{-1}\int \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\,\boldsymbol{j}(x)\cdot G(x, y; \chi)\boldsymbol{j}(y) - \tfrac{1}{2}\int \mathrm{d}\boldsymbol{x}\,\rho(x)(\chi(x) - m_0^2)$$
$$+\tfrac{1}{2}\hbar\beta\delta(0)\int \mathrm{d}\boldsymbol{x}\,\ln\rho(x) + \tfrac{1}{2}\hbar\,\mathrm{Tr}\ln(-\nabla^2 + \chi) \tag{3.6}$$

and

$$(-\nabla_x^2 + \chi(x))G(x, y; \chi) = \delta(x - y). \tag{3.7}$$

Since the individual terms in $\mathfrak{A}$ are all expected to be O(1) we see from (3.5) that the $1/N$ expansion for Z' (or, more appropriately, for the effective action Γ) is obtained by a saddle-point development of the path integral.

We have learnt (Ebbutt and Rivers 1982a) that the $1/N$ expansion uniquely accommodates the hard-core effects of the change of measure in an additive way. To leading order we need perform no integrations. Let the single extremum of $\mathfrak{A}$ be at $\rho = \rho_0[\boldsymbol{j}]$, $\chi = \chi_0[\boldsymbol{j}]$, whence

$$2\left.\frac{\delta\mathfrak{A}}{\delta\rho(x)}\right|_{\rho_0,\chi_0} = 0 = -\chi_0(x, [\boldsymbol{j}]) + m_0^2 + \frac{\hbar\beta\delta(0)}{\rho_0(x, [\boldsymbol{j}])} \tag{3.8}$$

$$2\left.\frac{\delta\mathfrak{A}}{\delta\chi(x)}\right|_{\rho_0,\chi_0} = 0 = \hbar G(x, x; \chi_0) + \boldsymbol{\varphi}_0(x, [\boldsymbol{j}])^2/N - \rho_0(x, [\boldsymbol{j}]) \tag{3.9}$$

where

$$(-\nabla_x^2 + \chi_0(x, [\boldsymbol{j}]))\boldsymbol{\varphi}_0(x, [\boldsymbol{j}]) = \boldsymbol{j}(x). \tag{3.10}$$

Then, for large N

$$W[\boldsymbol{j}] = -\hbar\ln Z'[\boldsymbol{j}] = N\mathfrak{A}[\rho_0[\boldsymbol{j}], \chi_0[\boldsymbol{j}], \boldsymbol{j}]. \tag{3.11}$$

To the same order the semiclassical fields $\bar{\boldsymbol{\varphi}}$ are

$$\bar{\varphi}_i(x) = -(\delta W/\delta j_i(x)) = \varphi_{0i}(x). \tag{3.12}$$

In consequence the effective action $\Gamma[\bar{\boldsymbol{\varphi}}]$ is

$$\Gamma[\bar{\boldsymbol{\varphi}}] = W[\boldsymbol{j}[\bar{\boldsymbol{\varphi}}]] + \int \mathrm{d}\boldsymbol{x}\,\boldsymbol{j}(x, [\bar{\boldsymbol{\varphi}}])\cdot\bar{\boldsymbol{\varphi}}(x) \tag{3.13}$$

$$= N\mathfrak{A}[\rho_0[\boldsymbol{j}[\bar{\boldsymbol{\varphi}}]], \chi_0[\boldsymbol{j}[\bar{\boldsymbol{\varphi}}]], \boldsymbol{j}[\bar{\boldsymbol{\varphi}}]]$$
$$+\int \mathrm{d}\boldsymbol{x}\,\bar{\boldsymbol{\varphi}}(x)[-\nabla_x^2 + \chi_0(x, [\boldsymbol{j}[\bar{\boldsymbol{\varphi}}]])]\bar{\boldsymbol{\varphi}}(x) + \mathrm{O}(N^0) \tag{3.14}$$

$$= \tfrac{1}{2}\int \mathrm{d}\boldsymbol{x}\{\bar{\boldsymbol{\varphi}}(x)[-\nabla_x^2 + \chi_0(x, [\boldsymbol{j}[\bar{\boldsymbol{\varphi}}]])]\bar{\boldsymbol{\varphi}}(x) - N\rho_0(x, [\boldsymbol{j}[\bar{\boldsymbol{\varphi}}]])[\chi_0(x, [\boldsymbol{j}[\bar{\boldsymbol{\varphi}}]]) - m_0^2]\}$$
$$+\tfrac{1}{2}\hbar N\beta\delta(0)\int \mathrm{d}\boldsymbol{x}\,\ln\rho_0(x, [\boldsymbol{j}[\bar{\boldsymbol{\varphi}}]]) + \tfrac{1}{2}\hbar N\,\mathrm{Tr}\ln(-\nabla^2 + \chi_0[\boldsymbol{j}[\bar{\boldsymbol{\varphi}}]]). \tag{3.15}$$

From (3.8), (3.9) and (3.10) we see that there is a simpler way to express $\Gamma[\bar{\varphi}]$. If we define $\Gamma[\bar{\varphi}, \chi, \rho]$ by

$$\Gamma[\bar{\varphi}, \chi, \rho] = \tfrac{1}{2} \int \mathrm{d}x[\bar{\varphi}(x)(-\nabla_x^2 + \chi(x))\bar{\varphi}(x) - N\rho(x)(\chi(x) - m_0^2)] + \tfrac{1}{2}\hbar N\beta\delta(0) \int \mathrm{d}x \ln \rho(x) + \tfrac{1}{2}\hbar N \operatorname{Tr} \ln(-\nabla^2 + \chi), \tag{3.16}$$

then $\Gamma[\bar{\varphi}]$ of (3.15) is given by

$$\Gamma[\bar{\varphi}] = \Gamma[\bar{\varphi}, \chi_0[\bar{\varphi}], \rho_0[\bar{\varphi}]] \tag{3.17}$$

where $\chi_0[\bar{\varphi}]$, $\rho_0[\bar{\varphi}]$ are determined by

$$\left.\frac{\delta\Gamma[\bar{\varphi}, \chi, \rho]}{\delta\chi}\right|_{\chi_0, \rho_0} = 0 = \left.\frac{\delta\Gamma[\bar{\varphi}, \chi, \rho]}{\delta\rho}\right|_{\chi_0, \rho_0}. \tag{3.18}$$

This follows from the observation that $\chi_0[j]$, $\rho_0[j]$ depend on j only through $\varphi_0[j] = \bar{\varphi}$.

Rather than work with the effective action Γ it is sufficient for the moment to restrict ourselves to an analysis of the effective potential $\mathcal{V}(\varphi)$, obtained from $\Gamma[\bar{\varphi}]$ on taking $\bar{\varphi}(x)$ to have the space–time *constant* value φ as

$$\Gamma[\varphi] = \mathcal{V}(\varphi)\left(\int \mathrm{d}x\right). \tag{3.19}$$

Again to leading order, we define

$$\mathcal{V}(\varphi, \chi, \rho) = \tfrac{1}{2}\chi\varphi^2 - \tfrac{1}{2}N\rho(\chi - m_0^2) + \tfrac{1}{2}\hbar N\beta\delta(0) \ln \rho + \tfrac{1}{2}\hbar N \int đk \ln(k^2 + \chi) \tag{3.20}$$

(where $đk = (2\pi)^{-d}\, \mathrm{d}^d k$). It follows that

$$\mathcal{V}(\varphi) = \mathcal{V}(\varphi, \chi_0(\varphi^2), \rho_0(\varphi^2)) \tag{3.21}$$

where

$$\chi_0(\varphi^2) = m_0^2 + \beta\hbar\delta(0)/\rho_0(\varphi^2) \tag{3.22}$$

$$\rho_0(\varphi^2) = \varphi^2/N + \hbar G(\chi_0(\varphi^2)) \tag{3.23}$$

with

$$G(\chi) = \int đk (k^2 + \chi)^{-1}. \tag{3.24}$$

In Ebbutt and Rivers (1982c) we examined the properties of $\mathcal{V}(\varphi)$ in the vicinity of $\varphi^2 = 0$ for general values of β. Our aim was to demonstrate that the addition of the term $\frac{1}{2}\hbar N\beta\delta(0) \ln(\varphi^2/N)$ to the classical potential via the change of measure (2.5) did not destabilise the effective potential at $\varphi^2 = 0$. In general we found that $\mathcal{V}(\varphi)$ was, after renormalisation, that of a free theory. In that paper we had assumed that it was the non-leading terms of the $1/N$ expansion that provided the non-trivial interactions. We no longer have this belief. In the remaining sections we examine the exceptions to the general case, not discussed in Ebbutt and Rivers (1982c), as candidates for non-trivial theories.

Finally, we note that, to leading order, the Euclidean and Minkowski effective potentials are identical. Thus, on the one hand we can interpret the effective potential

$\mathscr{V}(\varphi)$ as a vacuum energy density while, on the other hand, we can compare our results with the Euclidean statistical mechanics of Klauder (1981a, b).

4. Renormalisation in $d>4$ dimensions

In this section we examine the renormalisation of $\mathscr{V}(\varphi)$ in $d>4$ dimensions. Since $\mathscr{V}(\varphi, \chi, \rho)$ is extremised by χ_0, ρ_0 we see that (expressing $\mathscr{V}$ as a function of φ^2)

$$2\, \mathrm{d}\mathscr{V}(\varphi^2)/\mathrm{d}(\varphi^2) = \chi_0(\varphi^2). \tag{4.1}$$

Thus, to construct a renormalised $\mathscr{V}(\varphi^2)$ it is sufficient to obtain a renormalised $\chi_0(\varphi^2)$. Eliminating ρ_0 via (3.22) we have

$$\chi_0(\varphi^2) = m_0^2 + \frac{\beta\delta(0)}{G(\chi_0(\varphi^2)) + \varphi^2/\hbar N} \tag{4.2}$$

$$= m_0^2 + \frac{\beta\delta(0)}{G(\chi_0(\varphi^2))}\left(1 - \frac{\varphi^2}{\hbar N G(\chi_0(\varphi^2))} + \ldots\right). \tag{4.3}$$

As an intermediate step we regularise $\delta(0)$ and G by imposing an ultraviolet cut-off $|k| \leqslant \Lambda$. For simplicity we assume initially that β is *independent* of Λ. We define

$$\delta(0)_\Lambda = \int_{|k|\leqslant\Lambda} đk \tag{4.4}$$

$$G(\chi)_\Lambda = \int_{|k|\leqslant\Lambda} đk\,(k^2+\chi)^{-1} \tag{4.5}$$

and, for reasons that will become obvious,

$$B(\chi)_\Lambda = \int_{|k|\leqslant\Lambda} đk\,(k^2+\chi)^{-2} = -\frac{\partial G(\chi)_\Lambda}{\partial\chi}. \tag{4.6}$$

For $d>4$ dimensions

$$G(\chi)_\Lambda = G(0)_\Lambda - \chi B(0)_\Lambda + \chi^2 \int \mathrm{d}k\,(k^2)^{-2}(k^2+\chi)^{-1} \tag{4.7}$$

whence

$$\frac{\delta(0)_\Lambda}{G(\chi)_\Lambda} = \frac{\delta(0)_\Lambda}{G(0)_\Lambda} + \chi\frac{\delta(0)_\Lambda B(0)_\Lambda}{G(0)_\Lambda^2} + \mathrm{O}(\chi^2)$$

$$= \left(\frac{d-2}{d}\right)\Lambda^2 + \frac{\chi(d-2)^2}{d(d-4)} + \mathrm{O}(\Lambda^{-1}). \tag{4.8}$$

Inserting this expression in (4.3) it follows that $\chi_0(\varphi^2)$ satisfies

$$\chi_0(\varphi^2)\left(1 - \frac{\beta\delta(0)_\Lambda B(0)_\Lambda}{G(0)_\Lambda^2}\right) = \left(m_0^2 + \frac{\beta\delta(0)_\Lambda}{G(0)_\Lambda}\right) + \mathrm{O}(\Lambda^{-1}) + \frac{\varphi^2}{\hbar N}\mathrm{O}(\Lambda^{4-d}). \tag{4.9}$$

We see that the finite non-zero value of β,

$$\beta = \beta_0(d) = \frac{G(0)_\Lambda^2}{\delta(0)_\Lambda B(0)_\Lambda} = \frac{d(d-4)}{(d-2)^2}, \tag{4.10}$$

will have particular significance which we now assess.

4.1. *The case $\beta \neq \beta_0(d)$*

If $\beta \neq \beta_0(d)$, on removing the cut-off (i.e. $\Lambda \to \infty$) we have

$$\chi_0(\varphi^2) = m^2\beta_0(\beta_0 - \beta)^{-1} \tag{4.11}$$

where m_0^2 is chosen so that

$$m^2 = m_0^2 + \frac{\beta\delta(0)_\Lambda}{G(0)_\Lambda} = m_0^2 + \beta\left(\frac{d-2}{d}\right)\Lambda^2 \tag{4.12}$$

is finite.

That is, $\chi_0(\varphi^2)$ is *independent* of φ^2, whence from (4.1)

$$\mathscr{V}(\varphi^2) = \tfrac{1}{2}\chi_0\varphi^2 \tag{4.13}$$

corresponding to a trivial theory.

Thus, a necessary (but not sufficient) condition that we have a non-trivial theory is that in d dimensions β be fixed at the unique value $\beta_0(d)$.

4.2. *The case $\beta = \beta_0(d)$*

We need more detail than is present in (4.8), since any non-triviality will be a consequence of non-leading ultraviolet behaviour. Non-leading terms are given in detail in the appendix.

4.2.1. $d > 6$ dimensions, $\beta = \beta_0(d)$. From the appendix we see that equation (4.3) becomes

$$0 = \left[m_0^2 + \left(\frac{d-4}{d-2}\right)\Lambda^2\right] - \frac{4}{(d-4)(d-6)}\frac{\chi_0^2(\varphi^2)}{\Lambda^2} - \frac{\varphi^2}{\hbar N}\frac{(d-4)}{S_d}\Lambda^{4-d}(1 + \mathrm{O}(\Lambda^{-2})) + \mathrm{O}(\Lambda^{-3}). \tag{4.14}$$

Rewriting this as

$$\chi_0^2(\varphi^2) = \left[m_0^2 + \left(\frac{d-4}{d-2}\right)\Lambda^2\right]\Lambda^2(d-4)(d-6)$$
$$- \frac{\varphi^2}{\hbar N}\frac{(d-4)^2(d-6)}{4S_d}\Lambda^{6-d}(1 + \mathrm{O}(\Lambda^{-2})) + \mathrm{O}(\Lambda^{-1}) \tag{4.15}$$

we see that, in order to get a sensible expression as $\Lambda \to \infty$, we must keep

$$m^4 = \left[m_0^2 + \left(\frac{d-4}{d-2}\right)\Lambda^2\right]\Lambda^2 \tag{4.16}$$

finite in this limit. In consequence $\chi_0(\varphi^2) = m^2$ is independent of φ^2 as $\Lambda \to \infty$. Yet again we get the free-field result

$$\mathscr{V}(\varphi^2) = \tfrac{1}{2}\chi_0\varphi^2. \tag{4.17}$$

4.2.2. $d = 6$ dimensions, $\beta = \frac{3}{4}$. From the appendix we see that equation (4.3) now becomes

$$0 = (m_0^2 + \tfrac{1}{2}\Lambda^2) + \frac{\chi_0(\varphi^2)^2}{\Lambda^2}\ln\left(\frac{e^2\chi_0(\varphi^2)}{\Lambda^2}\right) - \frac{2}{S_6\Lambda^2}\frac{\varphi^2}{\hbar N} + \mathrm{O}(\Lambda^{-3}). \tag{4.18}$$

We rewrite this as

$$\chi_0(\boldsymbol{\varphi}^2)^2 = \frac{(m_0^2+\frac{1}{2}\Lambda^2)\Lambda^2}{[\ln(\Lambda^2/m^2)-\ln(e^2\chi_0(\boldsymbol{\varphi}^2)/m^2)]} - \frac{2}{S_6}\frac{\boldsymbol{\varphi}^2}{\hbar N}\frac{1}{[\ln(\Lambda^2/m^2)-\ln(e^2\chi_0(\boldsymbol{\varphi}^2)/m^2)]}. \tag{4.19}$$

To get a finite χ_0 as $\Lambda\to\infty$ we must vary m_0^2 so that $(m_0^2+\frac{1}{2}\Lambda^2)\Lambda^2(\ln\Lambda^2)^{-1}$ remains finite. As before, χ_0 is independent of $\boldsymbol{\varphi}^2$ in this limit to replicate the trivial result (4.16). Taking this and the previous result together we conclude that for $d\geqslant 6$ dimensions the large-N pseudofree theory is *always* trivial.

4.2.3. $d=5$ dimensions, $\beta=\frac{5}{9}$. From the appendix we see that there is a qualitative difference for $d=5$ dimensions in that the second non-leading term in the Λ series expansion of $\delta(0)_\Lambda/G(x)_\Lambda$ is $O(\Lambda^{-1})$, rather than $O(\Lambda^{-2})$. This is crucial for our result. From equation (4.3) we now get

$$0=(m_0^2+\tfrac{1}{3}\Lambda^2)-\frac{\pi}{2\Lambda}\chi_0(\boldsymbol{\varphi}^2)^{3/2}-\frac{12\pi^3}{\Lambda}\frac{\boldsymbol{\varphi}^2}{\hbar N}+O(\Lambda^{-2}). \tag{4.20}$$

If we vary m_0^2 as $\Lambda\to\infty$ so that

$$\varphi_0^2=\frac{N\hbar}{12\pi^3}\Lambda(m_0^2+\tfrac{1}{3}\Lambda^2) \tag{4.21}$$

remains finite, equation (4.20) becomes

$$\chi_0(\boldsymbol{\varphi}^2)^{3/2}=24(N\hbar)^{-1}\pi(\varphi_0^2-\boldsymbol{\varphi}^2) \tag{4.22}$$

as $\Lambda\to\infty$.

Thus, for $\boldsymbol{\varphi}^2\leqslant\varphi_0^2$ we have a *non-trivial* pseudofree effective potential, obtained from (4.1) via (4.22) as

$$\mathscr{V}(\boldsymbol{\varphi}^2)=\frac{3N}{10}\left(\frac{24\pi}{\hbar}\right)^{2/3}\left[\left(\frac{\varphi_0^2}{N}\right)^{5/3}-\left(\frac{\varphi_0^2}{N}-\frac{\boldsymbol{\varphi}^2}{N}\right)^{5/3}\right]. \tag{4.23}$$

This is shown schematically in figure 1. The minimum of $\mathscr{V}(\boldsymbol{\varphi}^2)$ is the symmetric point $\boldsymbol{\varphi}^2=0$.

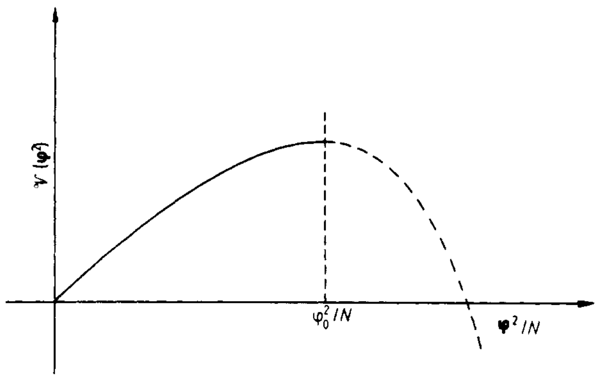

Figure 1. The effective potential $\mathscr{V}(\boldsymbol{\varphi}^2)$ for the large-N O(N)-invariant pseudofree theory in $d=5$ dimensions with $\beta=\beta_0(5)=\frac{5}{9}$. $\mathscr{V}(\boldsymbol{\varphi}^2)$ is real for $\boldsymbol{\varphi}^2\leqslant\varphi_0^2$, complex for $\boldsymbol{\varphi}^2>\varphi_0^2$. For $\boldsymbol{\varphi}^2>\varphi_0^2$ the dotted curve corresponds to Re $\mathscr{V}(\boldsymbol{\varphi}^2)$.

If we go back to the effective action $\Gamma[\boldsymbol{\varphi}^2]$ of (3.15) we find that

$$\frac{\delta^2\Gamma[\boldsymbol{\varphi}^2]}{\delta\varphi_i(x)\delta\varphi_j(y)}=[-\nabla_x^2+\chi_0(x,[\boldsymbol{\varphi}])]\delta(x-y)\delta_{ij}+\varphi_i(x)\frac{\delta\chi(x,[\boldsymbol{\varphi}])}{\delta\varphi_j(y)} \tag{4.24}$$

whence the common mass m of the φ fields is

$$m=\chi_0(0)^{1/2}=(24\pi\varphi_0^2/\hbar N)^{1/3}. \tag{4.25}$$

If we define the coupling strength of the theory by

$$\lambda=12\hbar N\frac{\mathrm{d}^2\mathscr{V}(\boldsymbol{\varphi}^2)}{\mathrm{d}(\boldsymbol{\varphi}^2)^2}\bigg|_{\boldsymbol{\varphi}^2=0} \tag{4.26}$$

we get

$$\lambda=-4\hbar\left(\frac{24\pi}{\hbar}\right)^{2/3}\left(\frac{\varphi_0^2}{N}\right)^{-1/3}=-\frac{96\pi}{m} \tag{4.27}$$

determined by m. We have a theory with non-zero φ–φ interaction.

For $\boldsymbol{\varphi}^2>\varphi_0^2$ we have to be careful. By $\chi^{3/2}$, in (4.20), we definitely mean $\chi\sqrt{\chi}$ with the positive square root. The real root on the right-hand side of (4.22) is therefore ruled out for $\boldsymbol{\varphi}^2>\varphi_0^2$. In consequence $\mathscr{V}(\boldsymbol{\varphi}^2)$ is complex for $\boldsymbol{\varphi}^2>\varphi_0^2$, a situation reminiscent of the *canonically* quantised self-interacting O(N)-invariant $\lambda N^{-1}(\boldsymbol{\varphi}^2)^2$ theory in $d=5$ dimensions in the large-N limit (Rembiesa 1978). We defer any further comments on this similarity until § 6.

We conclude with the observation that, from the above calculations, it seems almost accidental that there is a critical value of β for $d=5$, and no such values for $d>5$. The reason for this becomes more apparent when we examine the dynamics of the interaction later.

5. Renormalisation in $d=4$ dimensions

For $d=4$ space–time dimensions the expression (4.9) for $\beta_0(d)$ is invalid but it is straightforward to see that there is no *non-zero* critical value of β in the $\Lambda\to\infty$ limit. Unlike the case of $d>4$ dimensions we shall need to renormalise β if we are to get a non-trivial theory. In so far as we can interpret β as a coupling constant in (3.4) this is not unexpected.

Equation (4.3) now becomes (see appendix)

$$\chi_0(\boldsymbol{\varphi}^2)=m_0^2+\tfrac{1}{2}\beta_\Lambda[\Lambda^2-\chi_0(\boldsymbol{\varphi}^2)\ln(\chi_0(\boldsymbol{\varphi}^2)/\Lambda^2)]-8\pi^2\beta_\Lambda(\boldsymbol{\varphi}^2/\hbar N)+\mathrm{O}(\Lambda^{-2}) \tag{5.1}$$

where we have anticipated by the suffix Λ that β_Λ needs to be regularised in a Λ-dependent way.

We see that, if $\beta_\Lambda\to\beta_0\neq0$ as $\Lambda\to\infty$, $\chi_0(\boldsymbol{\varphi}^2)$ will be independent of $\boldsymbol{\varphi}^2$ and we will have a trivial theory. Let us assume that, for some b and M^2, we can parametrise β_Λ as

$$\beta_\Lambda(M)=[b/\ln(\Lambda^2/M^2)]; \tag{5.2}$$

then

$$\begin{aligned}\chi_0(\boldsymbol{\varphi}^2)=(m_0^2+\tfrac{1}{2}\beta_\Lambda\Lambda^2)+\tfrac{1}{2}b\chi_0(\boldsymbol{\varphi}^2)[1-\ln(\chi/M^2)[\ln(\Lambda^2/M^2)]^{-1}]\\-8\pi^2b(\boldsymbol{\varphi}^2/\hbar N)[\ln(\Lambda^2/M^2)]^{-1}+\mathrm{O}(\Lambda^{-2}).\end{aligned} \tag{5.3}$$

For general values of b we will continue to have a trivial theory. However, *for the case* $b=2$ equation (5.3) becomes

$$0=(m_0^2\ln(\Lambda^2/M^2)+\Lambda^2)-\chi_0(\varphi^2)\ln(\chi_0(\varphi^2)/M^2)-16\pi^2(\varphi^2/\hbar N)+\mathrm{O}(\Lambda^{-2}). \tag{5.4}$$

Thus, keeping

$$\mu_0^2=m_0^2\ln(\Lambda^2/M^2)+\Lambda^2 \tag{5.5}$$

finite as $\Lambda\to\infty$ we get

$$\chi_0(\varphi^2)\ln[\chi_0(\varphi^2)/M^2]+16\pi^2(\varphi^2/\hbar N)=\mu_0^2. \tag{5.6}$$

That is, χ_0 depends on φ^2 and we have a *non-trivial* pseudofree theory. At first this seems a little strange since, from (5.2), we see that

$$\beta_\infty=0 \tag{5.7}$$

whereas if we had set $\beta=0$ initially in (3.2) we would have had a free theory. There is no dichotomy. This is an example of discontinuous perturbations in which the 'hard core' weeds out configuration histories throughout the regularisation procedure†. Arguments are presented by Klauder (1981a) for β being suitably zero in $d=4$ dimensions.

In figure 2 we display equation (5.6) schematically. As in the case of $d=5$ dimensions discussed previously we have a value $\varphi_0^2>0$ such that $\mathscr{V}(\varphi^2)$ is complex for $\varphi^2>\varphi_0^2$. If χ_c is the value of χ at which $\varphi^2=\varphi_0^2$ then

$$\chi_c=M^2\mathrm{e}^{-1} \tag{5.8}$$

whence

$$\varphi_0^2=\frac{\hbar N}{16\pi^2}(\mu_0^2+\chi_c). \tag{5.9}$$

Furthermore, we see that $\mathscr{V}(\varphi^2)$ is real and double-valued for

$$\hbar N\mu_0^2/16\pi^2\leqslant\varphi^2\leqslant\varphi_0^2 \qquad \mu^2>0 \tag{5.10}$$

$$0\leqslant\varphi^2\leqslant\varphi_0^2 \qquad \mu^2<0. \tag{5.11}$$

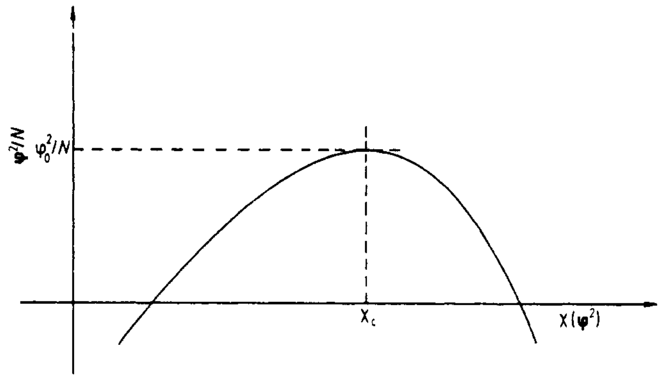

Figure 2. The relationship between $\chi_0(\varphi^2)$ and φ^2/N for $d=4$ dimensions with a regularised $\beta_\Lambda=2(\ln\Lambda^2/M^2)^{-1}$.

† A similar situation occurs in the quantum mechanics of singular potentials (Klauder 1979).

In figure 3 we display $\mathscr{V}(\varphi^2)$ schematically for $\mu^2<0$. The lower (upper) branch corresponds to $\chi>\chi_c$ ($\chi<\chi_c$). The physical branch is expected to be the lower†. The situation is remarkably like the large-N limit of the canonical $\lambda N^{-1}(\varphi^2)^2$ theory in $d=4$ dimensions (Kobayashi and Kugo 1975, Abbott *et al* 1976).

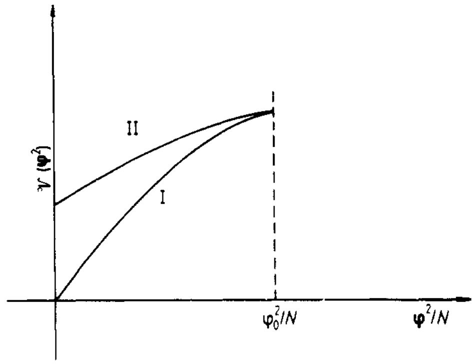

Figure 3. The effective potential $\mathscr{V}(\varphi^2)$ for the large-N O(N)-invariant pseudofree theory in $d=4$ dimensions with β as in figure 2 and $\mu_0^2<0$. The branches I and II are identified with the segments of $\chi_0(\varphi^2)$ in figure 2 that are similarly labelled. $\mathscr{V}(\varphi^2)$ is complex for $\varphi^2>\varphi_0^2$.

Let us now consider the coupling strength λ, of the theory. On the physical branch we have

$$\lambda(\eta)=12\hbar N\frac{\mathrm{d}^2\mathscr{V}(\varphi^2)}{\mathrm{d}(\varphi^2)^2}\bigg|_{\varphi^2=\eta^2}=\frac{-96\pi^2}{\ln(\chi(\eta^2)/\chi_c)} \tag{5.12}$$

non-zero and negative as in five dimensions.

The above is reminiscent of the *canonical* quantisation of a *self-interacting* $\lambda_0(\varphi^2)^2$ theory in $d=4$ dimensions in the large-N limit (Kobayashi and Kugo 1975, Abbott *et al* 1976). To see that the two formalisms are, in fact, *identical* we shall recapitulate the large-N limit of the O(N)-invariant theory with (Euclidean) Lagrangian density

$$\mathscr{L}=-\tfrac{1}{2}(\nabla\varphi)^2-\tfrac{1}{2}m_0^2\varphi^2-(\lambda_0/4!N)(\varphi^2)^2 \tag{5.13}$$

when quantised *canonically* (i.e. with $\beta\equiv0$).

We require both mass and coupling constant renormalisation. We define finite μ^2 and g by (Abbott *et al* 1976)

$$\begin{aligned}\frac{\mu^2}{g}&=\frac{m_0^2}{\lambda_0}+\tfrac{1}{6}\int\frac{\mathrm{d}k}{k^2}\\ \frac{1}{g}&=\frac{1}{\lambda_0}+\tfrac{1}{6}\int\frac{\mathrm{d}k}{k^2(k^2+M^2)}.\end{aligned} \tag{5.14}$$

If we take $\mu^2\to\infty$, $g\to\infty$ in such a way that

$$96\pi^2\mu^2g^{-1}=-\mu_0^2 \tag{5.15}$$

† We note that for $\mu_0^2>0$ the upper branch (5.10) corresponds to a spontaneous breaking of the symmetry to O($N-1$).

the effective potential $\mathcal{V}_c(\boldsymbol{\varphi}^2)$ for the *canonically* quantised theory (5.13) becomes identical to the effective potential $\mathcal{V}(\boldsymbol{\varphi}^2)$ for the *non-canonically* quantised *pseudofree* theory, obtained from (3.1)†.

That is, when β is tuned as (5.2) (with $b=2$) so as to give a non-trivial theory the hard-core effect of the change of measure (after renormalisation) is just that of a $\lambda_0(\boldsymbol{\varphi}^2)^2$ interaction (again after renormalisation)

With this increased understanding we now return to the $d>4$ dimensional case of § 4.

6. The case $d>4$ revisited

The previous section has shown the importance of regularising β in the cut-off theory. In § 4 we had neglected such a possibility. We now relax this condition, taking β_Λ to have a Λ dependence in $d>4$ dimensions so that

$$\lim_{\Lambda\to\infty}\beta_\Lambda=\beta>0. \tag{6.1}$$

We see immediately that, if $\beta\neq\beta_0(d)$ in d dimensions, the large-N pseudofree theory in d dimensions remains *free*. Thus, we can only hope to have a non-trivial pseudofree theory provided

$$\lim_{\Lambda\to\infty}\beta_\Lambda=\beta_0(d). \tag{6.2}$$

Even this will not enable us to get a non-trivial theory for $d>5$ dimensions. If, for example, for $d>6$ dimensions we take

$$\beta_\Lambda=\beta_0(d)[1+(M/\Lambda)^\alpha]\qquad \alpha>0 \tag{6.3}$$

where M is an arbitrary mass scale, equation (4.14) is replaced by

$$\chi_0(\boldsymbol{\varphi}^2)\left(\frac{M}{\Lambda}\right)^\alpha=\left[m_0^2+\left(\frac{d-4}{d-2}\right)\Lambda^2\right]-\frac{4}{(d-4)(d-6)}\frac{\chi_0(\boldsymbol{\varphi}^2)^2}{\Lambda^2}$$
$$-\frac{\boldsymbol{\varphi}^2}{\hbar N}\frac{(d-4)}{S_d}\Lambda^{4-d}+\mathrm{O}(\Lambda^{-3}). \tag{6.4}$$

For all α, $\chi_0(\boldsymbol{\varphi}^2)$ remains constant as before, because of the dominance of the Λ behaviour of the second term over the third term on the right-hand side of (6.4).

Although the presence of logarithms slightly complicates the case $d=6$ dimensions, we are equally unable to obtain non-trivial results in this case.

However, for $d=5$ dimensions the situation is different. If we take

$$\beta_\Lambda=\tfrac{5}{9}[1+(M/\Lambda)] \tag{6.5}$$

equation (4.20) is replaced by

$$-\frac{M}{\Lambda}\chi_0(\boldsymbol{\varphi}^2)=(m_0^2+\tfrac{1}{3}\Lambda^2+\tfrac{1}{3}M\Lambda)-\frac{\pi}{2\Lambda}\chi_0(\boldsymbol{\varphi}^2)^{3/2}-\frac{12\pi^3}{\Lambda}\frac{\boldsymbol{\varphi}^2}{\hbar N}+\mathrm{O}(\Lambda^{-2}). \tag{6.6}$$

† It is sufficient to establish the identity of $\chi_0(\boldsymbol{\varphi}^2)$ for the two cases, since $\partial\mathcal{V}/\partial(\boldsymbol{\varphi}^2)=\frac{1}{2}\chi_0$. The reader is referred to Abbott *et al* (1976) for more details.

Rewriting this as

$$\frac{1}{2}\frac{\varphi^2}{\hbar N}=-\frac{\mu^2}{g}-\frac{\chi_0(\varphi^2)}{48\pi^2}\left(\chi_0(\varphi^2)^{1/2}-\frac{4M}{\pi}\right) \tag{6.7}$$

where g, μ^2 are such that $g\to\infty$, $\mu^2\to\infty$ with

$$\frac{\mu^2}{g}=-\frac{\Lambda}{24\pi^3}(m_0^2+\tfrac{1}{3}\Lambda^2+\tfrac{1}{3}M\Lambda) \tag{6.8}$$

finite, we see that we recover the general case of the *canonical* $\lambda_0(\varphi^2)^2$ large-N theory in $d=5$ dimensions (Rembiesa 1978). This has a more complicated structure than the $d=4$ case, and we refer the reader to Rembiesa for further details.

We merely note that, for both $d=5$ and $d=4$ dimensions, the quantities μ^2/g of (5.15) and (6.8) have the same *renormalisation-invariant* definition (5.14) in terms of the bare parameters of the canonical theory. The quantity g is, however, *not* renormalisation invariant and hence the fact that the correspondence between the non-canonical pseudofree theory and the canonical $\lambda(\varphi^2)^2$ theory occurs at $g\to\infty$ in each case has no significance.

The fact that it is not possible to construct a non-trivial large-N renormalised $O(N)$-invariant $\lambda_0(\varphi^2)^2$ theory in $d\geqslant 6$ dimensions (which follows immediately from (5.14)) is the counterpart to our inability to construct a large-N non-trivial pseudofree theory in the same dimensions.

Finally, we observe that, as a large-N result, the equivalence between the non-canonical pseudofree theory and the canonical interacting theory (when they both exist) is essentially an equivalence of the most singular parts of the 'hard core' of the non-canonical theory and the self-interaction of the interacting canonical theory (Ebbutt and Rivers 1982b). Without examining the non-leading terms in detail, we do not know whether the less singular parts of the 'hard core' and the self-interaction are identical (i.e. whether the non-leading terms in the $1/N$ expansion agree).

7. Dynamics in $d\geqslant 4$ dimensions

Although the effective potential can determine the non-triviality of a pseudofree theory, to understand the nature of the φ self-interactions we must go back to the effective action $\Gamma[\varphi]$ of § 3, since it is Γ that contains the dynamics of the theory.

It is most convenient to work with the more general $\Gamma[\varphi,\chi,\rho]$ of (3.16), in which φ, χ, ρ are taken to be independent fields. The momentum-space matrix of inverse propagators, evaluated at the extremum $\varphi=\mathbf{0}$, $\chi=\chi_0(0)=m^2$, $\rho=\rho_0(0)$, has the form

$$\mathscr{D}(k^2)=\begin{pmatrix} k^2+m^2 & 0 \\ 0 & \mathbf{D}_{\chi\rho}(k^2)\end{pmatrix} \tag{7.1}$$

where $\mathbf{D}_{\chi\rho}$ is the 2×2 matrix (first row χ, second row ρ)

$$\mathbf{D}_{\chi\rho}(k^2)=-\tfrac{1}{2}N\begin{pmatrix} \hbar B(k^2,m^2) & 1 \\ 1 & \delta(0)\beta/\hbar G(m^2)^2\end{pmatrix} \tag{7.2}$$

with

$$B(k^2,m^2)=\int đp[(k-p)^2+m^2]^{-1}(p^2+m^2)^{-1}. \tag{7.3}$$

From (6.2) we see that the χ, ρ propagators are $O(1/N)$, whereas the ρ–χ mixing vertex is $O(N)$. Since the φ_i fields only interact directly with the χ field via a Yukawa interaction we see that the leading contribution to the φ–φ interaction is given by the Born diagrams of figure 4†.

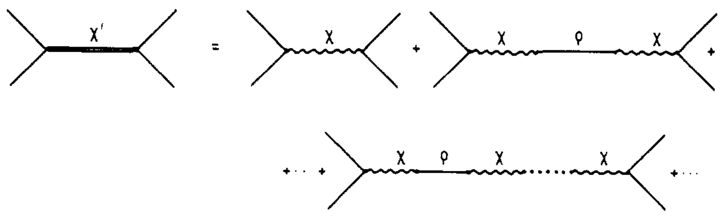

Figure 4. The Born diagrams contributing to φ–φ scattering in the large-N limit. Full lines denote external φ fields or internal ρ fields, wavy lines χ fields, and the double line the diagonalised χ' field.

On summing these diagrams we see that the φ–φ interaction takes place via the exchange of an $O(N)$ scalar field (χ', say) with propagator

$$D_{\chi'}(k^2) = -\frac{N\delta(0)\beta}{2G(m^2)^2}[\det \mathbf{D}_{\chi\rho}(k^2)]^{-1}$$

$$= \frac{2\delta(0)\beta}{NG(m^2)^2}\left(1 - \frac{\beta\delta(0)}{G(m^2)^2}B(k^2, m^2)\right)^{-1}. \tag{7.4}$$

In $d>4$ dimensions, for which $B(k^2=0,\ m^2=0)_\Lambda$ of (6.3) exists and is equal to $B(m^2=0)_\Lambda$ of (4.6), we can write the ultraviolet cut-off χ' propagator as

$$D_{\chi'}(k^2)_\Lambda = \frac{2}{N}\left(\frac{\beta}{\beta_0(d)}\right)\left(\frac{1}{B(0,0)_\Lambda}\right)\left(\frac{G(0)^2_\Lambda}{G(m^2_2)^2_\Lambda}\right)\left(1 - \frac{\beta}{\beta_0(d)}\frac{B(k^2, m^2)_\Lambda G(0)^2_\Lambda}{B(0,0)_\Lambda G(m^2)^2_\Lambda}\right)^{-1} \tag{7.5}$$

where we have used (4.10). For simplicity we have again assumed that β is *independent* of Λ.

We now understand the role of $\beta_0(d)$. If $\beta \neq \beta_0(d)$

$$D_{\chi'}(k^2)_\Lambda \sim \frac{2}{N}\frac{\beta(\beta_0-\beta)^{-1}}{B(0,0)_\Lambda} = O(\Lambda^{4-d}) \to 0 \qquad \text{as } \Lambda \to \infty. \tag{7.6}$$

That is, the χ' necessarily vanishes.

If $\beta = \beta_0(d)$ we have a more complicated situation with

$$D_{\chi'}(k^2)_\Lambda = \frac{2N^{-1}G(0)^2_\Lambda}{G(m^2)^2_\Lambda B(0,0)_\Lambda - G(0)^2_\Lambda B(k^2, m^2)_\Lambda}. \tag{7.7}$$

On expanding $G(m^2)_\Lambda$ as in (4.7) we see that

$$D_{\chi'}(k^2)_\Lambda = \frac{2N^{-1}}{[B(0,0)_\Lambda - B(k^2, m^2)_\Lambda] - 2m^2 B(0,0)^2_\Lambda G(0)^{-1}_\Lambda[1 + O(\Lambda^{-2})]}. \tag{7.8}$$

† The only other interactions present in (3.16) are χ self-interactions of order N, and ρ self-interactions of the same order. These give no contributions to leading order.

As we vary d we find the following.

(i) *$d>6$ dimensions* ($\beta \equiv \beta_0(d)$). In this case $B(k^2, m^2) = \mathrm{O}(\Lambda^{d-4})$, whence

$$[B(0,0)_\Lambda - B(k^2, m^2)_\Lambda] = \mathrm{O}(\Lambda^{d-6}) \tag{7.9}$$

diverges as $\Lambda \to \infty$, as does

$$B(0,0)^2_\Lambda G(0)^{-1}_\Lambda = \mathrm{O}(\Lambda^{d-6}) \tag{7.10}$$

with similar behaviour. Thus choosing $\beta = \beta_0(d)$ does not stop the χ' field vanishing, with $D_{\chi'}(k^2)_\Lambda = \mathrm{O}(\Lambda^{d-6})$.

(ii) *$d=6$ dimensions* ($\beta \equiv \frac{3}{4}$). The χ' propagator still vanishes for $\beta = \beta_0(d)$, albeit as an inverse logarithm.

(iii) *$d=5$ dimensions* ($\beta \equiv \frac{5}{9}$). For this dimension $[B(0,0)_\Lambda - B(k^2, m^2)_\Lambda]$ is *finite* whence, as $\Lambda \to \infty$, we have

$$D_{\chi'}(k^2) = \frac{2N^{-1}}{B(0,0) - B(k^2, m^2)}, \tag{7.11}$$

that is, the χ' does *not* vanish in this case, but propagates. The properties of χ' are obtained from ($k^2 = -s \geqslant -4m^2$)

$$B(-s, m^2)_\Lambda = \frac{1}{24\pi^3}\int_{4m^2}^{\Lambda^2} \frac{\mathrm{d}t}{t^{1/2}}\frac{(t-4m^2)}{(t-s)} = \frac{1}{24\pi^3}\left(\int_{4m^2}^{\Lambda^2}\frac{\mathrm{d}t}{t^{1/2}} + (s-4m^2)\int_{4m^2}^{\Lambda^2}\frac{\mathrm{d}t}{t^{1/2}(t-s)}\right) \tag{7.12}$$

whence

$$B(0,0) - B(-s, m^2) = \frac{1}{24\pi^3}\left(\int_0^{4m^2}\frac{\mathrm{d}t}{t^{1/2}} - (s-4m^2)\int_{4m^2}^{\infty}\frac{\mathrm{d}t}{t^{1/2}(t-s)}\right). \tag{7.13}$$

For $s<4m^2$ we see that both terms in (6.13) are positive and hence $D_{\chi'}(-s)$ has no pole below the elastic φ–φ threshold. For $s>4m^2$ equation (6.14) gives

$$D_{\chi'}(-s)^{-1} = \frac{N}{64\pi^2}\left[m + \frac{(4m^2-s)}{2\sqrt{s}}\left(\mathrm{i}\pi + \ln\left(\frac{\sqrt{s}+2m}{\sqrt{s}-2m}\right)\right)\right] \tag{7.14}$$

whence $\mathrm{Re}\, D_{\chi'}(-s)^{-1} < 0$ for large s. Thus, we have a two-ϕ *resonance* whose mass $M_{\chi'}$ satisfies the equation

$$x = (x-1)\ln\left(\frac{x+1}{x-1}\right) \qquad x = M_{\chi'}/2m \tag{7.15}$$

with solution

$$x>1 \qquad \Gamma_{\chi' \to 2\varphi} > 0. \tag{7.16}$$

How would the situation change if we allowed β to vary with Λ? (It follows immediately that there is no change for $d>5$ dimensions.) If we choose β_Λ as in (6.5) we find that

$$D_{\chi'}(k^2) = \frac{2N^{-1}}{[B(0,0)_\Lambda - B(k^2, m^2)_\Lambda] - (M/\Lambda)B(0,0)_\Lambda}. \tag{7.17}$$

As M varies χ' no longer need be a resonance but can become a bound state. The details are uninteresting and again the reader is referred to Rembiesa for details.

(iv) *$d=4$ dimensions.* Since $B(0,0)$ is infrared divergent in $d=4$ dimensions we need to modify our formulae. Introducing

$$B'(M^2)_\Lambda = \int_{|\boldsymbol{k}|\leq\Lambda} \mathrm{d}k\,(k^2)^{-1}(k^2+M^2)^{-1} \tag{7.18}$$

via

$$G(M^2)_\Lambda = G(0)_\Lambda - M^2 B'(M^2)_\Lambda \tag{7.19}$$

we see that

$$\beta_\Lambda(M) = [2/\ln(\Lambda^2/M^2)] \tag{7.20}$$

(the relevant β_Λ of § 5) satisfies

$$1 = \frac{\beta_\Lambda(M)\delta(0)_\Lambda B'(M^2)_\Lambda}{G(0)_\Lambda^2}. \tag{7.21}$$

Reverting to (7.4) it is straightforward to see that, if $\beta \neq \beta_\Lambda(M)$ for some M, $D_{\chi'}(k^2)_\Lambda = \mathrm{O}[(\ln \Lambda^2)^{-1}]$ to give a trivial theory. However, if we take $\beta = \beta_\Lambda(M)$ we find that

$$D_{\chi'}(k^2)_\Lambda = \frac{2N^{-1}G(0)_\Lambda^2}{G(m^2)_\Lambda^2 B'(M^2)_\Lambda - G(0)_\Lambda^2 B(k^2, m^2)_\Lambda}. \tag{7.22}$$

Using (7.20) it follows that, as $\Lambda \to \infty$,

$$D_{\chi'}(k^2) = \frac{2N^{-1}}{B'(M^2) - B(k^2, m^2)}, \tag{7.23}$$

a non-zero propagator showing the persistence of χ'.

On evaluating $B(k^2, m^2)$ (Abbott *et al* 1976) we find ($k^2 = -s \geq -4m^2$)

$$D_{\chi'}(-s)^{-1} = \frac{N}{16\pi^2}[\tfrac{1}{2}\ln(m^2/\chi_c) + (f(-s, m^2) - f(0, m^2))] \tag{7.24}$$

where $\chi_c = M^2 \mathrm{e}^{-1}$ was introduced in (5.7), and

$$f(-s, m^2) = \left(\frac{4m^2 - s}{s}\right)^{1/2} \ln\left(\frac{(4m^2-s)^{1/2} + s^{1/2}}{2m}\right). \tag{7.25}$$

Since

$$0 \leq [f(k^2, m^2) - f(0, m^2)] < \infty \qquad 0 \leq k^2 < \infty \tag{7.26}$$

tachyons will be absent from $D_{\chi'}(-s)$ if

$$m^2 = \chi_0(\varphi^2 = 0) > \chi_c. \tag{7.27}$$

This is true for the global ground state on branch I of $\mathscr{V}(\varphi^2)$ (see figures 2 and 3). Restricting ourselves to this state there are two possibilities.

(*a*) If $m^2 < \mathrm{e}^2\chi_c = \mathrm{e}M^2$, $\mathrm{Re}\, D_{\chi'}(-s)^{-1}$ has *two* zeros. One of these zeros corresponds to a *bound state* (denoted by χ'_1 with $M_{\chi'_1} < 2m$) and the other corresponds to an unstable *resonance* (denoted χ'_2, mass $M_{\chi'_2} > 2m$). In this case we can choose the φ-mass m and the bound-state mass $M_{\chi'_1}$ (or $M_{\chi'_2}$) as the two independent parameters determining the pseudofree theory. Given m each value of $M_{\chi'_1}$ gives a different theory.

(*b*) If $m^2 > e^2\chi_c = eM^2$ there is no bound state or resonance and the χ' system has no simple interpretation. The pseudofree theory is now determined uniquely by the choice of m and M.

8. Summary and interpretation

We have examined the large-N limit of O(N)-invariant pseudofree theories, labelled by a single (bare) parameter β_Λ that characterises the scale-covariant measure of the momentum cut-off regularised theory for which $|k| \leqslant \Lambda$. For general values of β we expect the large-N pseudofree theory to be a free theory.

In detail, we find, if $\beta_\Lambda \to \beta$ as $\Lambda \to \infty$, the following.

(i) We have a free theory for *all* values of β in $d \geqslant 6$ space–time dimensions.

(ii) We have a free theory in $d = 5$ space–time dimensions for all $\beta \neq \frac{5}{9}$. For $\beta = \frac{5}{9}$ we have a *non-trivial* theory. If $\beta_\Lambda = \frac{5}{9}(1 + M\Lambda^{-1})$ we have a different theory for each M. We understand this in the following way. For general values of β two φ's pass through one another without interaction. However, as we tune β to the value $\frac{5}{9}$, the two-φ system resonates to produce a bound-state or resonant O(N) scalar that we have called χ'. The properties of χ' are uniquely given by the φ mass. It is the exchange of χ' that provides the non-zero φ–φ interaction. The theory is determined uniquely on specifying the φ and χ' masses.

(iii) We have a free theory in $d = 4$ dimensions for all $\beta \neq 0$. However, it is possible to choose $\beta_\Lambda \neq 0$ so as to get a *non-trivial* theory as $\Lambda \to \infty$. In particular, we must take

$$\beta_\Lambda(M) = [2/\ln(\Lambda^2/M^2)] \tag{8.1}$$

where M is an arbitrary mass scale.

Although $\beta_\Lambda \to 0$ as $\Lambda \to \infty$ the pseudofree theory is not related to the free theory for which $\beta \equiv 0$ at all stages. The change of measure in the regularised theory is sufficient to enforce a discontinuous perturbation. Dynamically, what now happens is that the φ–φ system can simultaneously form a bound state, denoted χ'_1, and a resonance χ'_2 where β_Λ is given by (6.1). We can interchange M and $M_{\chi'_1}$ as arbitrary parameters. Thus, given m, we have a single-parameter family of (different) pseudofree theories labelled by the bound-state mass $M_{\chi'_1}$.

The φ–φ interaction then arises from the interchange of this bound-state–resonance complex. (It is also possible, when β_Λ is given by (6.1) for the two-φ fields to generate a non-resonant (or bound-state) system that can give φ–φ forces. The properties of this are completely determined by M.)

(iv) Whenever the large-N *non-canonical* pseudofree theory in $d = 4, 5$ dimensions is non-trivial it is *identical* to the large-N *canonically* quantised $\lambda_0(\varphi^2)^2$ theory. In so far as the large-N limit identifies the most singular part of the 'hard-core' interaction due to the change of measure (Ebbutt and Rivers 1982b) this shows that the most singular part of the non-canonical 'hard core' is equivalent to a 'canonical' $\lambda_0(\varphi^2)^2$ self-interaction, once renormalisation has been effected. Furthermore, the fact that the large-N limit in $d = 4, 5$ dimensions always has a symmetric vacuum shows that the pseudofree theory has a symmetric vacuum. This is anticipated by the nature of the change of measure with its origin in the unitary implementation of field scale transformations, and the non-unitary implementation of field translations.

Our first comment concerns the change of measure that non-canonical quantisation implies. As we mentioned earlier, an arbitrary change of measure would not lead to a consistent quantisation procedure. Despite the fact that a scale-covariant measure corresponds to perfectly acceptable (affine) commutation relations it is not obvious, *a priori*, that this is enough. In particular, our ability to construct a *unitary* theory is intimately related to the choice of measure. It follows from (6.11) and (6.23) that the large-N limit maintains unitarity (which, in this case, is essentially two-particle unitarity). This is because, after removing the cut-off, the χ' propagator (when it is non-trivial) is nothing more than a geometrical sum of φ 'bubbles'. Two-particle unitarity, automatically satisfied by such series (Zachariasen 1961), is all, given that the large-N limit is a 'tree' theory of φ's and χ''s. Moreover, from this viewpoint, we can anticipate how non-leading terms in a $1/N$ expansion would preserve unitarity. Similar arguments could be constructed for cluster decomposition.

The next general comment that we make is that our previous analysis (Ebbutt and Rivers 1982c) of the large-N pseudofree theory for $d<4$ dimensions is unchanged. That is, the only consistent pseudofree theory for $d<4$ dimensions is a *free* theory with $\beta\equiv 0$. That is, non-canonical quantisation is only appropriate for those dimensions for which canonical quantisation is over-restrictive in forcing the theory to be trivial. This is in accord with the heuristic argument of Klauder (1979), which used the Sobolev inequalities to suggest that the canonical translation-invariant measure be preserved for $d<4$ dimensions.

Our third observation concerns the high-temperature expansions of Klauder (1981a, b). Arguments were presented there that we should expect $\beta_0>0$ for $d>4$ dimensions and, in some sense, $\beta_0=0$ in $d=4$ dimensions. Our results agree with these conclusions for $d=5, 4$ dimensions. However, we note that in $d=4$ dimensions we not only needed $\beta\to 0$ but also that β_Λ was exactly as in (6.1) (e.g. $\beta_\Lambda=2.5(\ln \Lambda^2/M^2)^{-1}$ gives a trivial theory). This suggests that it may be very difficult to perform accurate numerical analysis in $d=4$ dimensions.

Fourthly, we stress that the existence of non-trivial pseudofree theories in the large-N limit does not, in itself, do anything to prove the existence of non-trivial scalar theories in $d>4$ dimensions. What we are saying here is that, in so far as non-trivial *non-canonical* scalar theories exist, they will need a functional measure that we expect to be mimicked well by the measures given here. The case for this is the following (Klauder 1981a). We are reminded that the existence of non-trivial *canonical* scalar theories in these dimensions is prevented by the coincidence of zero upper and lower bounds on the suitably defined connected four-point function. For the precontinuous *scale-covariant* lattice theory Klauder has argued that the upper bound, the Lebowitz (1974) inequality (where the suffix c denotes the connected part)

$$\langle\varphi_i\varphi_j\varphi_k\varphi_l\rangle_c\leqslant 0 \tag{8.2}$$

of canonical theory no longer holds. For example, the single-site φ-field moments satisfy (Klauder 1981a)

$$\langle\varphi^4\rangle_c/\langle\varphi^2\rangle^2=\beta(1-\beta)^{-1}>0 \qquad 0<\beta<1. \tag{8.3}$$

With the Lebowitz bound lifted, there is no known reason to prevent non-trivial scalar theories. If non-trivial theories do exist as a consequence of scale-covariant quantisation, it must be because they are based upon non-trivial pseudofree theories. For the reasons indicated earlier, this requires a φ–φ force to appear once β is correctly tuned. The mechanism we have described above, in which this force is (usually) a consequence of bound-state/resonance formation peculiar to these values of β, must be a recognisable caricature of the true dynamics.

Nonetheless, we conclude with a cautionary note. Although the dynamics of the φ–φ interaction (when there is any) is described by bound states and resonances with no tachyons or ghosts visible, the effective potentials $\mathcal{V}(\varphi^2)$ of figures 1 and 3 may have some pathologies for finite N. For example, in $d=5$ dimensions $\mathcal{V}(\varphi^2)$ of (4.21) is such that Re $\mathcal{V}(\varphi^2)$ becomes arbitrarily large and negative as $\varphi^2 \to \infty$. If tunnelling is able to render the $\varphi^2=0$ vacuum unstable for $N<\infty$ the tunnelling amplitude will be O(exp $-N$). Although this may not matter numerically for large finite N it does matter in principle. The situation for the analogous O(N)-invariant $N^{-1}\lambda(\varphi^2)^2$ theory has been looked at in some detail (Linde 1976, Cant 1979, Salomonson 1982). Although not understood completely, it would not be surprising if tunnelling did occur for this case, with the implication that the pseudofree theories presented here, if taken literally, would also suffer. (The proviso is that the tunnelling mechanism is via very non-classical instantons whose properties are not understood.) Even if this is the case we are not too deterred.

We draw an analogy with mean-field theory in statistical physics, to which the $1/N$ expansion has some similarity. Whereas the mean-field expansion can give a good approximation to the critical temperature (i.e. the critical coupling constant) but be in error over the nature of the transition we expect the $1/N$ expansion to give a good approximation to the critical value of β (itself a coupling constant) despite getting some global properties of $\mathcal{V}(\varphi^2)$ wrong. The compatibility of our results to those of Klauder (1981a) encourages this view.

There are two obvious next steps. Firstly, we need to reanalyse the scale-covariant quantisation of a classically self-interacting scalar theory for the critical values of β discussed here. The general result ($\beta \neq \beta_0(d)$) was given by Ebbutt and Rivers (1982d). Secondly, we must check the stability of our results under the inclusion of non-leading terms in the $1/N$ expansion. Both of these problems are under active consideration and the results will be published elsewhere.

Appendix. Basic formulae

Let

$$S_d = \frac{1}{(2\pi)^d}\omega_d = \frac{1}{(2\pi)^d}\frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{A1}$$

where ω_d denotes the surface area of the unit hypersphere in d dimensions.

A.1. $d>6$ space–time dimensions

$$\delta(0)_\Lambda/G(\chi)_\Lambda = \left(\frac{d-2}{d}\right)\Lambda^2 + \frac{(d-2)^2}{d(d-4)}\chi - \frac{4(d-2)^2}{d(d-4)^2(d-6)}\frac{\chi^2}{\Lambda^2} + \mathrm{O}(\Lambda^{-3}) \tag{A2}$$

$$\delta(0)_\Lambda/G(\chi)^2_\Lambda = \frac{(d-2)^2}{dS_d}\Lambda^{4-d} + \mathrm{O}(\Lambda^{2-d}). \tag{A3}$$

A.2. $d=6$ space–time dimensions

$$\delta(0)_\Lambda/G(\chi)_\Lambda = \tfrac{2}{3}\Lambda^2 + \tfrac{4}{3}\chi + \tfrac{4}{3}\chi^2\Lambda^{-2}\ln(\mathrm{e}^2\chi/\Lambda^2) + \mathrm{O}(\Lambda^{-3}) \tag{A4}$$

$$\delta(0)_\Lambda/G(\chi)^2_\Lambda = \tfrac{8}{3}S_6^{-1}\Lambda^{-2} + \mathrm{O}(\Lambda^{-4}). \tag{A5}$$

A.3. d = 5 space-time dimensions

$$\delta(0)_\Lambda/G(\chi)_\Lambda = \tfrac{3}{5}\Lambda^2 + \tfrac{9}{5}\chi - \tfrac{9}{10}\pi\Lambda^{-1}\chi^{3/2} + O(\Lambda^{-2}) \tag{A6}$$

$$\delta(0)_\Lambda/G(\chi)^2_\Lambda = \tfrac{9}{5}S_5^{-1}\Lambda^{-1} + O(\Lambda^{-3}) = \tfrac{108}{5}\pi^3\Lambda^{-1} + O(\Lambda^{-3}). \tag{A7}$$

A.4. d = 4 space-time dimensions

$$\delta(0)_\Lambda/G(\chi)_\Lambda = \tfrac{1}{2}\Lambda^2 - \tfrac{1}{2}\chi\ \ln(\chi/\Lambda^2) + O(\Lambda^{-2}) \tag{A8}$$

$$\delta(0)_\Lambda/G(\chi)^2_\Lambda = \frac{1}{S_4} + O(\Lambda^{-2}) = 8\pi^2 + O(\Lambda^{-1}). \tag{A9}$$

Unless specifically stated, logarithmic divergences are ignored in expressions $O(\Lambda^n)$.

References

Abbott L F, Kang J S and Schnitzer H J 1976 *Phys. Rev.* D **13** 2212
Aizenmann M 1981 *Phys. Rev. Lett.* **47** 1
Cant R J 1979 *Nucl. Phys.* B **157** 108
Ebbutt J M and Rivers R J 1982a *J. Phys. A: Math. Gen.* **15** 2911
—— 1982b *J. Phys. A: Math. Gen.* **15** 2947
—— 1982c *J. Phys. A: Math. Gen.* **15** 3273
—— 1982d *J. Phys. A: Math. Gen.* **15** 3285
Frohlich J 1982 *Nucl. Phys.* B **200** 281
Klauder J R 1978 *Science* **199** 735
—— 1979 *Ann. Phys.* **117** 19
—— 1981a *Phys. Rev.* D **24** 2599
—— 1981b High temperature expansions for non-renormalisable quantum field theories *Selected Topics in Quantum Field Theory and Mathematical Physics, Bechyne, Czechoslovakia* to be published
Kobayashi M and Kugo T 1975 *Prog. Theor. Phys.* **54** 1537
Lebowitz J L 1974 *Commun. Math. Phys.* **35** 87
Linde A D 1976 *Nucl. Phys.* B **125** 147
Rembiesa P 1978 *Phys. Rev.* D **17** 2830
Salomonson P 1982 *CERN preprint* TH 3281
Zachariasen F 1961 *Phys. Rev.* **121** 1851